Transformation Matrices – further development.

Paul Corrie FP1 assignment

A transformation in 2 dimensions can be said to map the point (x, y) onto its image (x', y'). If the transformation is repeated on the new point then (x', y') maps onto (x'', y'').

- What transformations are there where $(x, y) = (x'', y'')$?
- What pairs of transformations are there (2 different transformations) where the original point is mapped back on to itself?

Investigating types of transformation

Enlargement

1. If a shape is enlarged by scale factor s, centre $(0,0)$ then the transformation matrix $\mathbf{E}$ is $\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}$

- Find the image of (x, y) under transformation $\mathbf{E}$.
- By observation, what is the transformation matrix $\mathbf{E}^{-1}$, the inverse of $\mathbf{E}$?
- Find $\mathbf{EE}^{-1}$ and state what your result shows.

2. Enlargements can have different scale factors for x and y (more commonly called a two-way stretch) s_x and s_y.

- Find the transformation matrix $\mathbf{S}$ for this type of enlargement.
- Find $\mathbf{S}^{-1}$ and verify that $\mathbf{SS}^{-1} = \mathbf{I}$.

Shear

A shear transformation is where all the points along a given line (often an axis) remain fixed while other points are shifted parallel to the fixed line by a distance proportional to their distance from the fixed line.

- Find $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ and explain what effect each matrix has had.
- Find the inverse of each matrix.
- Verify, using a multiplication equaling $\mathbf{I}$, that you have found the inverses.

Rotation

An anticlockwise rotation $\mathbf{C}$, centre $(0,0)$ through an angle θ, has the transformation matrix:

$$\mathbf{C} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

- Using the fact that $\cos(-\theta) = \cos\theta$ and that $\sin(-\theta) = -\sin\theta$ find the inverse matrix $\mathbf{C}^{-1}$.
- Verify that $\mathbf{CC}^{-1} = \mathbf{I}$

Reflection

A reflection $\mathbf{R}$ in the line $y = mx$ has the transformation matrix

$$\mathbf{R} = \frac{1}{1+m^2}\begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}$$

- Showing each stage of your working, find $\mathbf{R}^2$ and explain your result.

Affine transformations

An affine transformation is one which consists of a combination of a linear transformation and a translation. To define a 2 dimensional translation in vector form it is useful to think of a 3 dimensional matrix (3 dimensional computer animators therefore need to work in 4 dimensions!).

Translation

To find the image of (x,y) after a translation t_x and t_y. the following matrix multiplication is used:

$$\begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x+t_x \\ y+t_y \\ 1 \end{pmatrix}$$

Notice the dummy row along the bottom to allow the matrices to be multiplied (this is called making them homogeneous).

- Write down the inverse of the transformation matrix.
- Showing your working, confirm that the matrix and its inverse have the product $\mathbf{I}$.

Rotation about the point (p,q) through angle θ

Geometrically this can be thought of as a sequence of 3 transformations:

1. A translation $\mathbf{T}_1$ of $\begin{pmatrix} -p \\ -q \end{pmatrix}$
2. A rotation $\mathbf{O}$, centre $(0,0)$ through angle θ,
3. A translation $\mathbf{T}_2$ of $\begin{pmatrix} p \\ q \end{pmatrix}$

Using the previous results find the matrices $\mathbf{T}_1$, $\mathbf{O}$ and $\mathbf{T}_2$

- For a rotation of 90° anticlockwise, centre $(3,5)$, confirm that $\mathbf{T}_1\mathbf{O}\mathbf{T}_2$ is $\begin{pmatrix} 0 & -1 & 8 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}$
- Use this matrix to confirm that the point $(4,7)$ is mapped onto $(1,6)$.

Reflection in the line y=mx+c

In a similar way to rotations about a point other than the origin, a reflection in the line $y = mx + c$ can be thought of as a sequence of 3 transformations:

1. A translation $\mathbf{T}_1$ of $\begin{pmatrix} 0 \\ -c \end{pmatrix}$
2. A reflection $\mathbf{Q}$ in the line $y = mx$
3. A translation $\mathbf{T}_2$ of $\begin{pmatrix} 0 \\ c \end{pmatrix}$

- For a reflection in the line $y = 2x+3$ find the matrices $\mathbf{T}_1$, $\mathbf{Q}$ and $\mathbf{T}_2$
- Use these results to find the single transformation matrix $\mathbf{P}$ for a reflection in the line $y = 2x+3$.
- Use your matrix to confirm that the points $(4,5)$ and $(-3,2)$ are reflected onto $(0.8, 7.4)$ and $(1,0)$ respectively.
- Confirm that $\mathbf{P}^2 = \mathbf{I}$.
- Find the single transformation matrix $\mathbf{P}$ in terms of m and c.