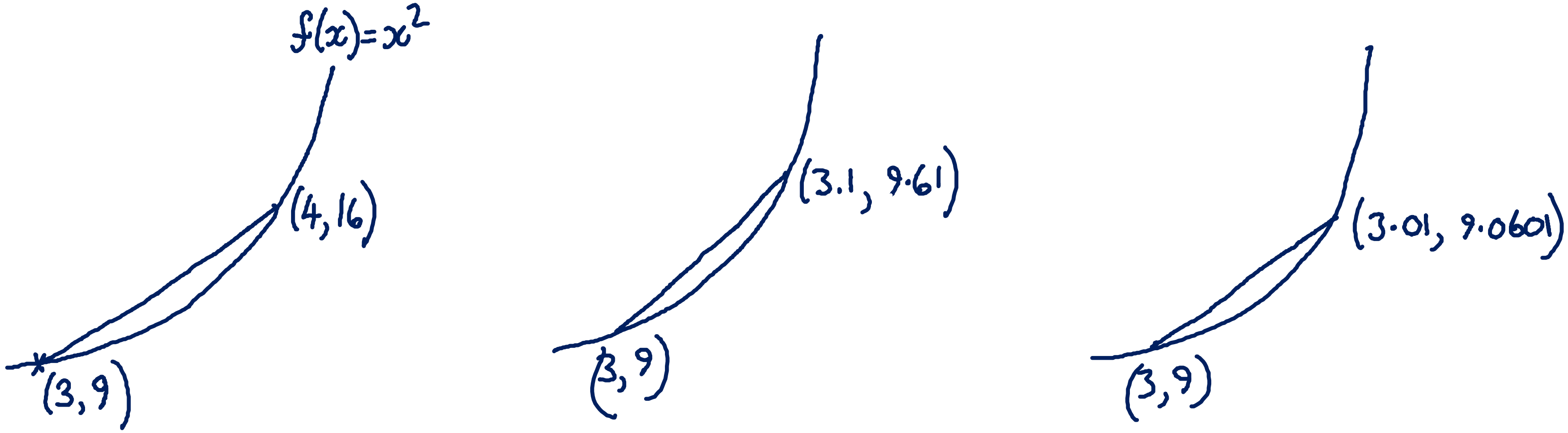
PROOF IN CALCULUS: Some thoughts ...

Differentiation from First Principles - from Day 1

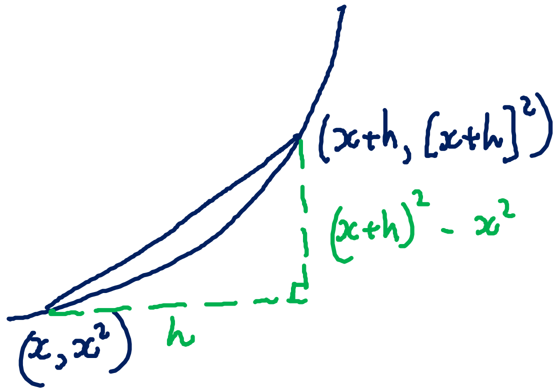
My path through this:

1. Drawing tangents, measuring gradients, spotting patterns
   - start with pencil, ruler and PRINTED graphs
   - perhaps move on to using technology to look at more graphs
2. Steps towards proof - look at the limit of chords



* SET UP A SPREADSHEET *

3. Formal proof:



$$\text{Gradient} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} 2x + h$$

$$= 2x$$

4. Generalise:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Try $f(x) = x^3$

Use knowledge of Binomial to generalise – able students will do this themselves with a little nudging if neccessary

Moving on ...

Differentiate each new function from first principles as you introduce it

e can be defined in terms of its derivative: the function for with $f'(x) = f(x)$

$\sin x$ Start by sketching the derivative to see it's a vertical stretch of $\cos x$

Use autograph to play around to get radians

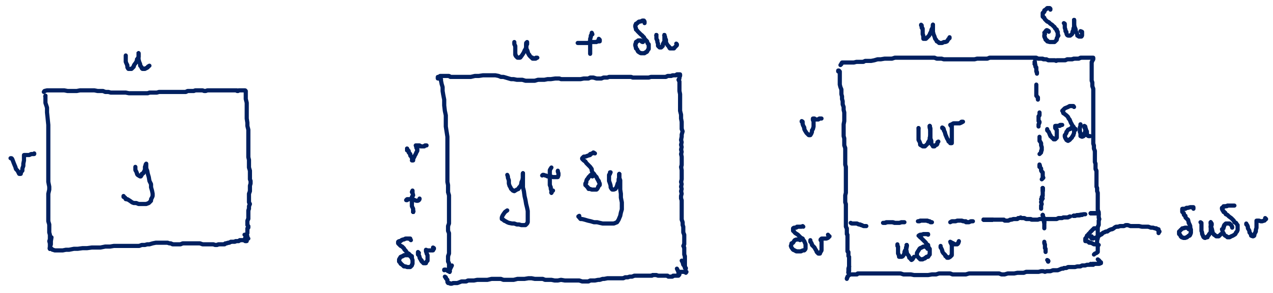
Use a spreadsheet to investigate the limit of $\frac{\sin x}{x}$ (this is the handwaving bit)

Then differentiate from first principles

Product Rule

- Before chain rule because it's easier to apply ?
- Check $\frac{d(uv)}{dx} \neq \frac{du}{dx} \times \frac{dv}{dx}$ for a polynomial function

$y = uv$



So $\delta y = u\delta v + v\delta u + \delta u \delta v$

$$\Rightarrow \frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \frac{\delta u}{\delta x}\delta v$$

As $\delta x \to 0$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} + \frac{du}{dx} \times 0$$

Quotient Rule – use $y = \frac{u}{v} \Rightarrow vy = u$ and use product rule

Chain Rule.

Sausage machine Input $x \rightarrow u \rightarrow y$ Output

Increase x by small amount δx

so u increases by δu

y increases by δy

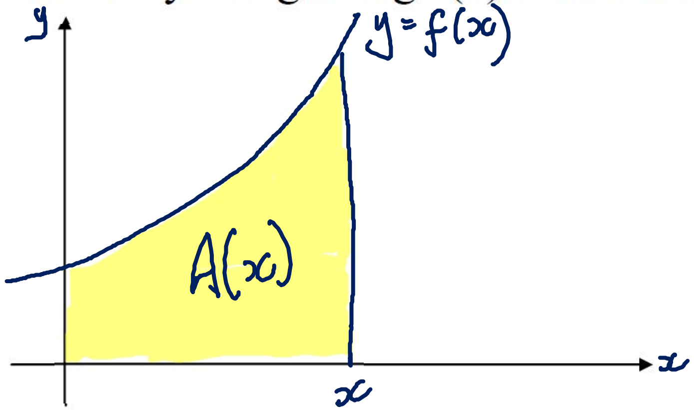
Then $\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}$ This is just fractions at this stage

As $\delta x \to 0$ we get $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Hmm... how do we make sure students are comfortable with δx etc?

FUNDAMENTAL THEOREM OF CALCULUS

The fact that the area under the graph of $y = \mathrm{f}(x)$ is found by integrating $\mathrm{f}(x)$ is known as the 'fundamental theorem of calculus'.

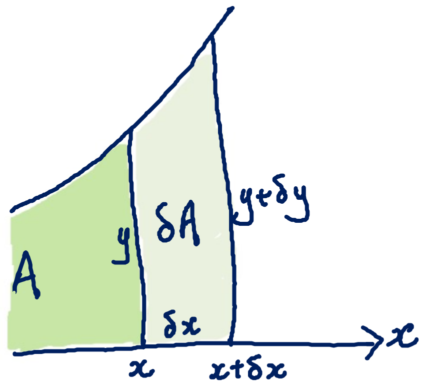


Put another way, the theorem says that

if A is the area under the graph of $y = \mathrm{f}(x)$
(measured from some starting value of x),

then $\dfrac{\mathrm{d}A}{\mathrm{d}x} = \mathrm{f}(x)$

To get an idea of why the theorem is true, think what happens when the value of x is increased by a small amount δx.



Let δy be the corresponding increase in y, and δA the increase in the area (shaded lighter). The extra area is very close to being a trapezium whose parallel sides are

So $\delta A = \frac{1}{2}(y+y+\delta y)\delta x = \left(y+\frac{1}{2}\delta y\right)\delta x$

So $\dfrac{\delta A}{\delta x} = y + \frac{1}{2}\delta y$

Now think what happens as δx gets smaller and smaller: δy also gets smaller and smaller, so $y + \frac{1}{2}\delta y$ gets closer to y, and the ratio $\dfrac{\delta A}{\delta x}$ gets closer to $\dfrac{\mathrm{d}A}{\mathrm{d}x}$.

So $\dfrac{\mathrm{d}A}{\mathrm{d}x} = y = \mathrm{f}(x)$