GAUSSIAN PRIMES INVESTIGATION

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INTRODUCTION

Prime numbers are often called the building blocks of mathematics, though it is probably fair to say that when they are introduced in primary or early secondary school they are seen as no more than another series of numbers (though with no pattern to help remember them).

Prime numbers in themselves are straightforward, though they can lead to some complex mathematics, some proven - eg the Prime Number Theorem – and some as yet unproven – eg Goldbach's conjecture. Fame and fortune await the mathematician who can turn this from a conjecture to a theorem!

PRIME NUMBERS AND COMPLEX NUMBERS

Once we venture into the world of complex numbers, prime numbers become a little harder to find.

- If **p = 3 +2j** and **q = 3 – 2j** show that the product **pq** is 13.
- Find another 3 complex factor pairs with the same product.

Consequently 13 is not a prime number once we allow complex numbers to be used.

GAUSSIAN PRIMES AND GAUSSIAN INTEGERS

In the mathematics of complex numbers a ***Gaussian prime number*** p can be defined as a number which has only 8 factors (written in pairs for clarity):

p	and	**1**
-p	and	**-1**
pj	and	**-j**
-pj	and	**j**

A ***Gaussian integer*** is similar to a real integer though with an imaginary part. It is written as **a + bj**, where **a** and **b** are integers.

- By finding extra factors over and above these 8, show that the following Gaussian integers are not Gaussian primes:
  - 2 (2 further factor pairs)
  - 5 (4 further factor pairs)
  - 3 + j (4 further factor pairs)

You will find it useful for the first 2 questions to think of $2 = 1^2 + 1^2$, and $5 = 1^2 + 2^2$ and to think of the difference of two squares.

- Check your answers for the questions above and try to find a method for splitting 13 and 17 into pairs of complex factors.

You should have noticed that 13 and 17 are both the sum of two squares, and that consequently by writing pairs of complex conjugates it is relatively straightforward finding the 4 complex factor pairs for each number.

- Investigate other real primes of the form of 4k + 1 (5, 13, 17, 29, 37….) and determine whether they are also Gaussian primes.
- Investigate real primes of the form 4k + 3 (3, 7, 11, 19, 23…..) and determine whether they are Gaussian primes.

GAUSSIAN PRIME FACTORISATION

To determine whether a Gaussian integer **a + bj** can be factorised it is useful to use the Norm of the number, or the distance from the origin on the Argand diagram.

To factorise **5 + 3j** we can use the following method:

Norm(**5 + 3j**) = $\sqrt{(5^2 + 3^2)} = \sqrt{34} = \sqrt{(2 \times 17)}$ and as we have found 2 = (1 + j)(1 – j) and 17 = (4 + j)(4 - j)

so Norm(**5 + 3j**) = $\sqrt{[(1 + j)(1 - j)(4 + j)(4 - j)]}$ **and (1 + j)(4 – j) = 5 + 3j** and *(1 – j)(4 + j) = 5 – 3j*, which is the complex conjugate

So we have found that **5 + 3j** is not a Gaussian prime as it has the factors **1 + j** and **4 – j**.

Use this method (or a simpler method in one case!) to investigate whether the following Gaussian integers are also Gaussian primes:

- 7 + 7j
- 7 + 6j
- 7 + 5j
- 7 + 4j
- 7 + 3j
- 7 + 2j
- 7 + j

GAUSSIAN 'SQUARE NUMBERS'

Choose Gaussian integers where a and b are coprime (no common factors except 1) and square them.

For example $(4 + 5j)^2 = -9 + 40j$

Investigate further and see what you notice.

Can you tell by observation whether the following are 'Gaussian squares', and if so find their square root?

- -3 + 4j
- 7 + 24j
- 6 – 8j
- 13 + 84j

GAUSSIAN PRIMES INVESTIGATION – Teaching notes

PRIME NUMBERS AND COMPLEX NUMBERS

This section is fairly straightforward, though the ability to multiply complex numbers is essential

$(3 + 2j)(3 - 2j) = 9 + 6j - 6j + 4 = 13$

Students should spot that this is the difference of two squares.

The other factor pairs of 13 pcorrie Page 3 4/16/2009(other than the Gaussian prime number factor pairs) are:

$(-3 + 2j)(-3 - 2j)$ $(2 + 3j)(2 - 3j)$ $(-2 + 3j)(-2 - 3j)$

GAUSSIAN PRIMES AND GAUSSIAN INTEGERS

Students need to realise that there will be some trial and error in these questions and that they need to pay particular attention to the signs:

$2 = (1 + j)(1 - j) = (-1 + j)(-1 - j)$
$5 = (1 + 2j)(1 - 2j) = (-1 + 2j)(-1 - 2j) = (2 + j)(2 - j) = (-2 + j)(-2 - j)$
$3 + j = (1 + j)(2 - j) = (1 - j)(1 + 2j) = (-1 + j)(-1 - 2j) = (-1 - j)(-2 + j)$
Students should not get too bogged down in the last question, as its point is to show that trial and error is not very efficient.

Students should realise that $13 = 3^2 + 2^2$ and that $17 = 4^2 + 1^2$ and use this to find that
$13 = (3 + 2j)(3 - 2j)$ and that
$17 = (4 + j)(4 - j)$ and all the other combinations as in the previous questions.

All primes of the form $4k + 1$ can be written as the sum of two square numbers, $m^2 + n^2$, and consequently are not Gaussian primes. Students should be guide towards the result that primes of this form can be factorised in the form
$(m + nj)(m - nj)$ and $(-m + nj)(-m - nj)$

All primes of the form $4k + 3$ are also Gaussian primes.

GAUSSIAN PRIME FACTORISATION

Once again there may need to be a little trial and error to pick the correct factor pairs that give the original Gaussian integer.

$7 + 7j = 7(1 + j)$ so not prime
$7 + 6j = (2 + j)(4 + j)$ so not prime
$7 + 5j = (1 - j)(1 + 6j)$ so not prime
$7 + 4j = (1 + 2j)(3 - 2j)$ so not prime
$7 + 3j = (1 + j)(5 - 2j)$ so not prime
$7 + 2j$ is a Gaussian prime: $7^2 + 2^2 = 53$ which is a prime of the form $4k + 1$, so $7 + 2j$ is also a Gaussian prime
$7 + j = (1 + j)(1 - 2j)(2 + j)$ - this may need some explanation (and some substantial trial and error):

$$7^2 + 1^2 = 50 = 2 \times 5 \times 5 = \boldsymbol{(1 + j)}(1 - j)(1 + 2j)\boldsymbol{(1 - 2j)(2 + j)}(2 - j)$$

It may be worth asking students to investigate the conjugates, as they will find that the factors of the conjugate of **a - bj** are the conjugates of the factors of **a + bj**

GAUSSIAN 'SQUARE NUMBERS'

Students should notice that they are dealing with Pythagorean triples. It may be useful to guide them to the set of triples:

(3, 4, 5)	(5, 12, 13)	(7, 24, 25)	(8, 15, 17)
(9, 40, 41)	(11, 60, 61)	(12, 35, 37)	(13, 84, 85)
(16, 63, 65)	(20, 21, 29)	(28, 45, 53)	(33, 56, 65)
(36, 77, 85)	(39, 80, 89)	(48, 55, 73)	(65, 72, 97)

$-3 + 4j = (1 + 2j)^2 = (-1 - 2j)^2$

$7 + 24j = (4 + 3j)^2$

$6 - 8j = 2(3 - 4j) = 2(2 - j)^2$, so not a Gaussian square

$13 + 84j = (6 + 7j)^2$