**Y13 A Level Mathematics**

**26 Differentiation 1.5 weeks**

## Teaching objectives

**A To review differentiation from AS, including finding maxima and minima**

**B To understand and use the second derivative in connection to convex and concave sections of curves and points of inflection**

**C Apply differentiation to find points of inflection**

**D Differentiate using the product rule**

**E Differentiate using the quotient rule**

**F Differentiate using the chain rule, including problems involving connected rates of change and inverse functions**

Note: In MEI SOW, this comes before differentiating advanced functions such as ln, cos, sin, ex etc and thus all resources used in this section need to exclude these functions and just consider polynomials. These rules will need to be revisited in the context of exponential and trig functions

**Resources for advance preparation: (Most can just be viewed electronically on students’ own devices or on teacher smartboard)**

**Only free resources linked below.**

[**UM: Two way calculus**](https://undergroundmathematics.org/calculus-meets-functions/two-way-calculus)– 1 sheet per student

[**Chain rule practice**](http://www.colmanweb.co.uk/Assets/Resources/ChainProductQuotient/ChainRulePractice.docx)– 1 sheet per student

[**KM**](http://kangaroomaths.com/free_resources/ks5/resources/session9/productandquotient/diffgame/) **Product & Quotient game –** game board and laminated sheets in pairs

[**Tangents and normals**](http://mei.org.uk/files/sow/25-differentiation-res.pdf) **from Integral** – one per pair

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | **Starter** | **Main teaching**Including key questions, key teaching points, models and resources | **Notes**Including Support and Extension | **Consolidation/Plenary**Including key questions and homework |
| **1** | To recap differentiation[**UM: R7391**](https://undergroundmathematics.org/calculus-of-powers/r7391) | **LO: Recap of AS - using differentiation to find stationary points and classify as maxima and minima***Discussion*: What is a stationary point? Why is it named so? Ask students to give examples of curves with different types of stationary points.How can we find and classify stationary points on curves? (AS recap)[**UM: Floppy Hair**](https://undergroundmathematics.org/calculus-of-powers/floppy-hair)Practises calculus of powers, finding maximum point, curve sketching (for missing curve) | Ensure students fully simplify polynomials before differentiating as, at this stage, have not learned product rule! | **Cons/Plenary**[**UM: Curvy cubics**](https://undergroundmathematics.org/calculus-meets-functions/can-you-find-curvy-cubics)**Home Learning**Routine practice from old resources (legacy C1/C2/AS) according to need. |
|  | **Starter** | **Main teaching**Including key questions, key teaching points, models and resources | **Notes**Including Support and Extension | **Consolidation/Plenary**Including key questions and homework |
| **2**  | [**UM: Gradient of gradients**](https://undergroundmathematics.org/calculus-meets-functions/gradients-of-gradients)(potentially just the first part – sketching the second derivative)  | **LO: Understand and use the second derivative in connection to convex and concave sections of curves and points of inflection. Apply differentiation to find points of inflection.**[**Geogebra: Shapes of curves**](https://www.geogebra.org/m/uwVATKud)Students should use their own tablet/laptop (if possible) to explore how the tangent to the curve changes and the impact on the derivatives. Summarise with class discussion; how can the second derivative give us information about the shape of a curve, even if the first derivative is not zero. This is the crucial new part that is different from AS to be reinforced. Students can then work in pairs on the following task which is to reinforce that we need to consider how the gradient of the curve is changing i.e. the rate of change of the gradient. **In addition**, ask students to describe the shape of the curve (eg convex upwards, non-stationary point of inflection etc) at the different points indicated.[**UM: R7433**](https://undergroundmathematics.org/calculus-of-powers/r7433)*Summary Example:* Give students the equation of a polynomial and ask them to sketch it, find and describe stationary points and points of inflection, and describe the shape of curve for different parts of the domain. | Ensure appropriate vocabulary is used; convex, concave, upwards vs downwards, stationary point, non-stationary point, point of inflection. Could get students to create a glossary.Khan Academy [**videos**](https://www.khanacademy.org/math/ap-calculus-ab/ab-derivatives-analyze-functions/ab-concavity/v/concavity-concave-upwards-and-concave-downwards-intervals) on convexity, points of inflection and practice questions*Extension/link to other topics:* Link to the normal distribution curve – probability density function has two points of inflection at ± one standard deviation from the mean. | **Cons/Plenary** (in pairs according to ability)[**Integral: Properties of curves**](https://2017.integralmaths.org/pluginfile.php/36039/mod_book/chapter/1050/Curves_Shapes.pdf)(not free)or [**concavity**](https://www.khanacademy.org/math/ap-calculus-ab/ab-derivatives-analyze-functions/ab-concavity/e/recognizing_concavity)from khan academy**Home Learning**[**Integral exercise level 1**](https://2017.integralmaths.org/pluginfile.php/36882/mod_resource/content/0/ocra2d1ax_level1.pdf)Or MyMaths/Y2/Shapes of Functions (but includes some e and ln) |

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | **Starter** | **Main teaching**Including key questions, key teaching points, models and resources | **Notes**Including Support and Extension | **Consolidation/Plenary**Including key questions and homework |
| **3** | Review of L2 home learning – students to share answers with class, check all happy before proceeding[**This**](https://www.khanacademy.org/math/ap-calculus-ab/ab-derivatives-analyze-functions/ab-concavity/e/analyze-points-of-inflection-graphical) question from Khan Academy | **LO: Understand and use the second derivative in connection to convex and concave sections of curves and points of inflection. Apply differentiation to find points of inflection.**[**UM: Two way calculus**](https://undergroundmathematics.org/calculus-meets-functions/two-way-calculus)(Resource in action: includes teacher support – see UM) | *Note:* Could combine lessons 2 and 3 into a single lesson depending on ability / pace of class*Note*: Although this has not been covered thoroughly on the A Level in recent years and hence legacy resources are a little limited, there are some older text books that contain good explanations eg Pure Mathematics 2 by Bostock & Chandler on p414-415*Extension:* refer to sufficient and necessary conditions for a point to be an inflection point (link to mathematical terminology and proof) – see [this](http://mathworld.wolfram.com/InflectionPoint.html) page from Wolfram Mathworld | **Cons/plenary**In small groups, students to create a glossary of terminology in table form including diagrams and examples to illustrate meanings. Use A3 paper and give students time to share ideas with other groups. Students can use devices to take a photo of the final versions.**Home Learning**[Integral exercise level 2](https://2017.integralmaths.org/pluginfile.php/36883/mod_resource/content/0/ocra2d1ax_level2.pdf)  |

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | **Starter** | **Main teaching**Including key questions, key teaching points, models and resources | **Notes**Including Support and Extension | **Consolidation/Plenary**Including key questions and homework |
| **4** | Several questions from legacy resources / text books to practice finding composite functions, including backwards questions eg find functions f(x) and g(x) such that fg(x) = (2x+1)3 or similar.Or [**these**](https://www.khanacademy.org/math/ap-calculus-ab/ab-derivative-rules/ab-chain-rule/e/identify-composite-functions) questions from Khan Academy | **LO: Differentiate using the chain rule**[**UM: Slippery slopes - another derivative**](https://undergroundmathematics.org/chain-rule/slippery-slopes-another-derivative)Introduces considering the derivative of composite functions by thinking about translations – allows students to discover chain rule for themselves. See teaching notes for key questions. *(could use this for starter if class already confident with composite functions)*Then introduce the chain rule formally, both versions i.e. [fg(x)]’ = f’[g(x)]g’(x) and dy/dx = dy/du x du/dx.[Integral: the chain rule](https://2017.integralmaths.org/pluginfile.php/36048/mod_book/chapter/1052/Diff_Chain.pdf)(not a free resource – sort cards to cut up and show the process of finding the derivative of four functions)*Particular examples to cover:* Contextual problems - e.g. the area of a circle is decreasing at a rate of 0.5ms-1. Find the rate of change of the radius when the radius is 2cm.[**UM: R7115**](https://undergroundmathematics.org/chain-rule/r7115) and [**UM: R6959**](https://undergroundmathematics.org/calculus-of-powers/r6959) are other contextual examples that could be usedInverse functions: Give students a fairly straightforward function and ask them to find its inverse, then differentiate both functions. This will (hopefully with prodding) allow them to deduce that dy/dx = 1 ÷ dx/dy. Link this to the geometry of inverse functions i.e. that the inverse function is a reflection in the line y = x and that the gradient at a point (a, b) on f(x) is the reciprocal of the gradient at the point (b, a) on f-1(x) which they (hopefully) will already be familiar with from their study of inverse functions. Practice: from legacy resources / textbooks, using this free exercise [**Chain rule practice**](http://www.colmanweb.co.uk/Assets/Resources/ChainProductQuotient/ChainRulePractice.docx) or using Integral exercise level 1 BUT ensure question selection includes some problems in context.  | *Support*Maths centre videos available for students needing extra recap at home[**Chain rule video**](file:///C%3A%5CUsers%5Calexa%5CDownloads%5Cmathcentre.ac.uk%5Cvideo%5C405)Several detailed Khan academy videos[**KA Chain rule videos**](https://www.khanacademy.org/math/ap-calculus-ab/ab-derivative-rules/ab-chain-rule/v/chain-rule-introduction)Could use Desmos or Geogebra here to graph both function and its inverse together with tangents at corresponding points to see relationship between gradients clearly. *Extension*: Integral exercise level 3 (chain rule)*Note*: Depending on available timing, you may wish to take two lessons for the chain rule (one to establish and practice, one for application to connected rates of change and inverse functions), or revisit in more detail once product and quotient rules have been established. Lesson 7 gives an opportunity to pull everything together and reinforce contextual and geometric applications. | **Plenary**[**UM: Can you find – chain rule edition**](https://undergroundmathematics.org/chain-rule/can-you-find-chain-rule-edition)(note – will need to leave out second question as students won’t have covered differentiating ln x yet)**Home Learning**Integral exercise level 1 / 2 (differentiate according to ability) or practice from legacy resources.Or MyMaths/Y2/Chain Rule(but includes some e and ln)Or MyMaths/Y2/Differentiating inverse functions (but contains some trig) |
| **5** | Recap / practice differentiating from first principles: Ask students to work in pairs (poss. on white boards) to differentiate y = x2 and then y = 2x(x2 -1) using limits | **LO: Differentiate using the product rule**Introduce product rule with the proof using limits (note – you may wish to introduce rule without proof at this stage and leave proof as an extension, depending on the class). Refer Khan Academy video if unsure of proof.Students to check their answer to the starter using the product rule and then complete another question (use legacy resources) to confirm use of rule.Combining product and chain rules Ask students to work in pairs to differentiate $f\left(x\right)= x^{2}\sqrt{3x-1}$. Emphasise the importance of setting work out very clearly; complete the chain rule part as a side calculation then link in clearly to the main part of the working. Practice: There are lots of legacy resources in textbooks etc on this to practice the techniques, or integral subscribers can use the product rule questions from integral exercise level 1 | *Support*Khan academy [**product rule videos**](https://www.khanacademy.org/math/ap-calculus-ab/ab-derivative-rules/ab-product-rule/v/applying-the-product-rule-for-derivatives)(includes examples of application and proof!)Powerpoint showing alternate derivation of product rule from [**Kangaroo Maths**](http://kangaroomaths.com/free_resources/ks5/resources/session9/productandquotient/derivingprodandquotrules.ppt)*Extension*[**UM R9082**](https://undergroundmathematics.org/product-rule/r9082) Finding a repeated root, opportunity for proof | **Plenary**Get students to close all books and notes and write down the formula for the product rule on a mini white board**Home Learning**Practice from legacy resources (ensure that includes some example where students have to use chain rule in conjunction with the product rule)Or MyMaths/Y2/Product Rule(but includes some e and ln) |

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | **Starter** | **Main teaching**Including key questions, key teaching points, models and resources | **Notes**Including Support and Extension | **Consolidation/Plenary**Including key questions and homework |
| **6** | Students to differentiate y = 2x/(x+1) by writing it as 2x(x+1)-1 and using the product rule. | **LO: Differentiate using the quotient rule (and choosing appropriate rule)**Students to establish quotient rule by writing u/v as uv-1 and using the product rule (note, this will require a very careful application of the chain rule in the general case – weaker students will need support with this)Once the quotient rule has been established, students to check their answer to the starter. NB Students may ask why they can’t just always use the product rule by writing a quotient as uv-1. Emphasise that subsequent algebraic manipulation and simplification is often made much easier by using the quotient rule. [Integral resources Activity 1:](https://2017.integralmaths.org/pluginfile.php/36056/mod_book/chapter/1054/Diff_Types.pdf) Types of Differentiation. (not a free resource) Students to sort expressions into a 3 way Venn re how they would differentiate; product vs quotient vs chain and then differentiate as many as they can (will need to leave out differentiating expressions involving e and ln etc for now but can still sort into Venn diagram)OrProduct and quotient rule differentiation game from [**Kangaroo Maths**](http://kangaroomaths.com/free_resources/ks5/resources/session9/productandquotient/diffgame/) (you will need to take out cards involving exponentials and trig)Or Practice from legacy text books. | *Support*Khan academy [**quotient rule videos**](https://www.khanacademy.org/math/ap-calculus-ab/ab-derivative-rules/ab-derivtive-rules-proofs/v/quotient-rule-from-product-rule), includes derivation of quotient rule from product and chain rules.Pair strong with weak for support*Extension*[**UM R9391:**](https://undergroundmathematics.org/product-rule/r9391) Combines with curve sketching and problem solving | **Plenary**“Mark my homework” style problem. Teacher prepared solution to a quotient rule problem containing deliberate errors for students to identify and correct. Could give a problem with harder to spot errors to stronger students.OrThe [**quotient rule song**](http://kangaroomaths.com/free_resources/ks5/resources/session9/productandquotient/quotientrulesong.doc) from Kangaroo Maths!**Home Learning**Questions from legacy resources or integral exercise level 2Or MyMaths/Y2/Quotient Rule |

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | **Starter** | **Main teaching**Including key questions, key teaching points, models and resources | **Notes**Including Support and Extension | **Consolidation/Plenary**Including key questions and homework |
| **7** | [**Risp 38: Differentiation Rules OK**](http://webfronter.com/waltham-forest/CFSMaths/mnu5/Rich_Starting_Points/images/risps_ebook_apr_07.pdf)(but will need to change the functions as this involves exp) | **LO: To use the chain, product and quotient rules in contextual and geometric problems (including connected rates of change) and identify which is the most appropriate.**Free resource from Integral[**Tangents and normals**](http://mei.org.uk/files/sow/25-differentiation-res.pdf)Contextual chain rule problems from lesson 4 if not already covered then.Consolidation and practice using legacy text book resources – ensure include a selection of all types.[This section is under construction on undergroundmathematics so very limited selection of free resources other than from legacy text books!] | *Extension*Integral exercise level 3 (product and quotient rules) | **Plenary**Review of entire topic – students to RAG rate different aspects**Home Learning**Integral Topic AssessmentOr MyMaths/Y2/Rates of Change |